

An extension of a Bourgain–Lindenstrauss–Milman inequality

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Abstract

Let $\|\cdot\|$ be a norm on \mathbb{R}^n . Averaging $\|(\varepsilon_1 x_1, \dots, \varepsilon_n x_n)\|$ over all the 2^n choices of $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, +1\}^n$, we obtain an expression $|||x|||$ which is an unconditional norm on \mathbb{R}^n .

Bourgain, Lindenstrauss and Milman [3] showed that, for a certain (large) constant $\eta > 1$, one may average over ηn (random) choices of $\vec{\varepsilon}$ and obtain a norm that is isomorphic to $|||\cdot|||$. We show that this is the case for any $\eta > 1$.

1 Introduction

Let $(E, \|\cdot\|)$ be a normed space, and let $v_1, \dots, v_n \in E \setminus \{0\}$. Define a norm $|||\cdot|||$ on \mathbb{R}^n :

$$|||x||| = \mathbb{E} \left\| \sum \varepsilon_i x_i v_i \right\|, \quad (1)$$

where the expectation is over the choice of n independent random signs $\varepsilon_1, \dots, \varepsilon_n$. This is an *unconditional* norm; that is,

$$|||(x_1, x_2, \dots, x_n)||| = |||(|x_1|, |x_2|, \dots, |x_n|)|||.$$

The following theorem states that it is sufficient to average $O(n)$, rather than 2^n , terms in (1), in order to obtain a norm that is isomorphic to $|||\cdot|||$ (and in particular approximately unconditional).

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Theorem. Let $N = (1 + \xi)n$, $\xi > 0$, and let

$$\{\varepsilon_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq N\}$$

be a collection of independent random signs. Then

$$\mathbb{P} \left\{ \forall x \in \mathbb{R}^n \quad c(\xi) \|x\| \leq \frac{1}{N} \sum_{j=1}^N \left\| \sum_{i=1}^n \varepsilon_{ij} x_i v_i \right\| \leq C(\xi) \|x\| \right\} \geq 1 - e^{-c'\xi n},$$

where

$$c(\xi) = \begin{cases} c\xi^2, & 0 < \xi < 1 \\ c, & 1 \leq \xi < C'' \\ 1 - C'/\xi^2, & C'' \leq \xi \end{cases}, \quad C(\xi) = \begin{cases} C, & 0 < \xi < C'' \\ 1 + C''/\xi^2, & C'' \leq \xi \end{cases},$$

and $c, c', C, C', C'' > 0$ are universal constants (such that $1 - C'/C''^2 \geq c$, $1 + C'/C''^2 \leq C$).

This extends a result due to Bourgain, Lindenstrauss and Milman [3], who considered the case of large ξ ($\xi \geq C''$); their proof makes use of the Kahane–Khinchin inequality. Their argument yields the upper bound for the full range of ξ , so the innovation is in the lower bound for small ξ .

With the stated dependence on ξ , the corresponding result for the scalar case $\dim E = 1$ was proved by Rudelson [6], improving previous bounds on $c(\xi)$ in [4, 1, 2]; see below. This is one of the two main ingredients of our proof, the second one being Talagrand’s concentration inequality [8] (which, as shown by Talagrand, also implies the Kahane–Khinchin inequality).

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2 Proof of Theorem

Let us focus on the case $\xi < 1$; the same method works (in fact, in a simpler way) for $\xi \geq 1$.

Denote $\|x\|_N = \frac{1}{N} \sum_{j=1}^N \left\| \sum_{i=1}^n \varepsilon_{ij} x_i v_i \right\|$; this is a random norm depending on the choice of ε_{ij} . Let $S_{\|\cdot\|}^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ be the unit sphere

of $(\mathbb{R}^n, ||| \cdot |||)$; we estimate

$$\begin{aligned} & \mathbb{P} \left\{ \forall x \in S_{|||\cdot|||}^{n-1}, c\xi^2 \leq |||x|||_N \leq C \right\} \\ & \geq 1 - \mathbb{P} \left\{ \exists x \in S_{|||\cdot|||}^{n-1}, |||x|||_N > C \right\} \\ & \quad - \mathbb{P} \left\{ \left(\forall y \in S_{|||\cdot|||}^{n-1}, |||y|||_N \leq C \right) \wedge \left(\exists x \in S_{|||\cdot|||}^{n-1}, |||x|||_N < c\xi^2 \right) \right\} . \end{aligned} \quad (2)$$

Upper bound: Let us estimate the first term

$$\mathbb{P} \left\{ \exists x \in S_{|||\cdot|||}^{n-1}, |||x|||_N > C \right\} .$$

Remark. As we mentioned, the needed estimate follows from the argument in [3]; for completeness, we reproduce a proof in the similar spirit.

Theorem (Talagrand [8]). *Let $w_1, \dots, w_n \in E$ be vectors in a normed space $(E, \|\cdot\|)$, and let $\varepsilon_1, \dots, \varepsilon_n$ be independent random signs. Then for any $t > 0$*

$$\mathbb{P} \left\{ \left| \left\| \sum_{i=1}^n \varepsilon_i w_i \right\| - \mathbb{E} \left\| \sum_{i=1}^n \varepsilon_i w_i \right\| \right| \geq t \right\} \leq C_1 e^{-c_1 t^2 / \sigma^2}, \quad (3)$$

where $c_1, C_1 > 0$ are universal constants, and

$$\sigma^2 = \sigma^2(w_1, \dots, w_n) = \sup \left\{ \sum_{i=1}^n \varphi(w_i)^2 \mid \varphi \in E^*, \|\varphi\|^* \leq 1 \right\} .$$

Remark. Talagrand has proved (3) with the median $\text{Med} \left\| \sum_{i=1}^n \varepsilon_i w_i \right\|$ rather than the expectation; one can however replace the median by the expectation according to the proposition in Milman and Schechtman [5, Appendix V].

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, denote

$$\sigma^2(x) = \sigma^2(x_1 v_1, \dots, x_n v_n) .$$

Claim 1. σ is a norm on \mathbb{R}^n and $\sigma(x) \leq C_2 |||x|||$ for any $x \in \mathbb{R}^n$.

Proof. The first statement is trivial. For the second one, note that

$$|||x||| = \mathbb{E} \left\| \sum \varepsilon_i x_i v_i \right\| \geq \mathbb{E} \left| \varphi \left(\sum \varepsilon_i x_i v_i \right) \right| = \mathbb{E} \left| \sum \varepsilon_i \varphi(x_i v_i) \right|, \quad \|\varphi\|^* \leq 1 .$$

Now, by the classical Khinchin inequality,

$$\sqrt{\sum y_i^2} \geq \mathbb{E} \left| \sum \varepsilon_i y_i \right| \geq C_2^{-1} \sqrt{\sum y_i^2} \quad (4)$$

(see Szarek [7] for the optimal constant $C_2 = \sqrt{2}$). Therefore

$$\|x\| \geq C_2^{-1} \sup_{\|\varphi\|^* \leq 1} \sqrt{\sum \varphi(x_i v_i)^2} = C_2^{-1} \sigma(x) .$$

□

By the claim and Talagrand's inequality, for every (fixed) $x \in S_{\|\cdot\|}^{n-1}$

$$\mathbb{P} \left\{ \left\| \sum_{i=1}^n \varepsilon_i x_i \right\| \geq t \right\} \leq C_1 \exp(-c_2 t^2) .$$

Together with a standard argument (based on the exponential Chebyshev inequality), this implies (for t large enough):

$$\mathbb{P} \left\{ \frac{1}{N} \sum_{j=1}^N \left\| \sum_{i=1}^n \varepsilon_{ij} x_i \right\| \geq t \right\} \leq \exp(-c_3 t^2 N) .$$

In particular, for $t = C_3 \geq \sqrt{4/c_3}$ the left-hand side is smaller than $12^{-N} < 6^{-n} 2^{-N}$.

The following fact is well-known, and follows for example from volume estimates (cf. [5]).

Claim 2. *For any $\theta > 0$, there exists a θ -net \mathcal{N}_θ with respect to $\|\cdot\|$ on $S_{\|\cdot\|}^{n-1}$ of cardinality $\#\mathcal{N}_\theta \leq (3/\theta)^n$.*

For now we only use this for $\theta = 1/2$. By the above, with probability greater than $1 - 2^{-N}$, we have: $\|x\|_N \leq C_3$ simultaneously for all $x \in \mathcal{N}_{1/2}$.

Representing an arbitrary unit vector $x \in S_{\|\cdot\|}^{n-1}$ as

$$x = \sum_{k=1}^{\infty} a_k x^{(k)}, \quad |a_k| \leq 1/2^{k-1}, \quad x^{(k)} \in \mathcal{N}_{1/2} ,$$

we deduce: $\|x\|_N \leq 2C_3$, and hence finally:

$$\mathbb{P} \left\{ \exists x \in S_{\|\cdot\|}^{n-1}, \|x\|_N > C \right\} \leq 2^{-N} \quad (5)$$

(for $C = 2C_3$).

Lower bound: Now we turn to the second term

$$\mathbb{P} \left\{ \left(\forall y \in S_{||\cdot||}^{n-1}, |||y|||_N \leq C \right) \wedge \left(\exists x \in S_{||\cdot||}^{n-1}, |||x|||_N < c\xi^2 \right) \right\} .$$

For σ_0 (that we choose later), let us decompose $S_{||\cdot||}^{n-1} = U \uplus V$, where

$$U = \left\{ x \in S_{||\cdot||}^{n-1} \mid \sigma(x) \geq \sigma_0 \right\}, \quad V = \left\{ x \in S_{||\cdot||}^{n-1} \mid \sigma(x) < \sigma_0 \right\} .$$

Recall the following result (mentioned in the introduction); we use the lower bound that is due to Rudelson [6].

Theorem ([4, 1, 2, 6]). *Let $N = (1 + \xi)n$, $0 < \xi < 1$, and let*

$$\{\varepsilon_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq N\}$$

be a collection of independent random signs. Then

$$\mathbb{P} \left\{ \forall y \in \mathbb{R}^n \quad c_4 \xi^2 |y| \leq \frac{1}{N} \sum_{j=1}^N \left| \sum_{i=1}^n \varepsilon_{ij} y_i \right| \leq C_4 |y| \right\} \geq 1 - e^{-c'_4 \xi n} ,$$

where $c_4, c'_4, C_4 > 0$ are universal constants, and $|\cdot|$ is the standard Euclidean norm.

Remark. By the Khinchin inequality (4), this is indeed the scalar case of Theorem 1 for $0 < \xi < 1$.

Thence with probability $\geq 1 - e^{-c'_4 \xi n}$ the following inequality holds for all $x \in U$ (simultaneously):

$$\begin{aligned} |||x|||_N &\geq \frac{1}{N} \sum_{j=1}^N \left| \varphi \left(\sum_{i=1}^n \varepsilon_{ij} x_i v_i \right) \right| = \frac{1}{N} \sum_{j=1}^N \left| \sum_{i=1}^n \varepsilon_{ij} x_i \varphi(v_i) \right| \\ &\geq c_4 \xi^2 \sigma(x) \geq c_4 \xi^2 \sigma_0 . \end{aligned} \tag{6}$$

Now let us deal with vectors $x \in V$. Let \mathcal{N}_θ be a θ -net on $S_{||\cdot||}^{n-1}$ (where θ will be also chosen later). For $x' \in \mathcal{N}_\theta$ such that $|||x - x'||| \leq \theta$, $\sigma(x') \leq \sigma_0 + C_2 \theta$ by Claim 1. Therefore by Talagrand's inequality (3),

$$\mathbb{P} \left\{ \left\| \sum_{i=1}^n \varepsilon_i x'_i v_i \right\| < 1/2 \right\} \leq C_1 \exp(-c_1 / (4(\sigma_0 + C_2 \theta)^2)) ,$$

and hence definitively

$$\begin{aligned} \mathbb{P} \left\{ \frac{1}{N} \sum_{j=1}^N \left\| \sum_{i=1}^n \varepsilon_{ij} x'_i v_i \right\| < 1/4 \right\} &\leq 2^N \left\{ C_1 \exp \left(-\frac{c_1}{4(\sigma_0 + C_2\theta)^2} \right) \right\}^{N/2} \\ &= \exp \left\{ - \left(\frac{c_1}{8(\sigma_0 + C_2\theta)^2} - \log(2\sqrt{C_1}) \right) N \right\} . \end{aligned}$$

Let $\sigma_0 = C_2\theta$, and choose $0 < \theta < 1/(8C)$ so that

$$\frac{c_1}{32C_2^2\theta^2} - \log(2\sqrt{C_1}) > \log 2 + \log(3/\theta) .$$

Then the probability above is not greater than $2^{-N}(\theta/3)^N < 2^{-N}/\#\mathcal{N}_\theta$ (by Claim 2). Therefore with probability $\geq 1 - 2^{-N}$ we have:

$$\|x'\|_N \geq 1/4 \quad \text{for } x' \in \mathcal{N}_\theta \text{ such that } \|x - x'\| < \theta \text{ for some } x \in V .$$

Using the upper bound (5), we infer:

$$\begin{aligned} \|x\|_N &\geq \|x'\|_N - \|x' - x\|_N \\ &\geq 1/4 - C/8C = 1/4 - 1/8 = 1/8 , \quad x \in V . \end{aligned} \tag{7}$$

The juxtaposition of (2), (5), (6), and (7) concludes the proof. \square

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